6.1 Number Theory

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- The study of the properties of counting numbers is called *number theory*.

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- The numbers 1, 2, 3, … are called the *counting numbers* or *natural numbers*.
- The study of the properties of counting numbers is called *number theory*.
- One interesting question is "What counting numbers can be written as a product of *other* numbers, and which cannot?"

If a and b are natural numbers, we say

$a \mid b$

to mean "a divides b" and it means that there is a number q with $b = a q$

We say that 5 divides 30 because there is a natural number 6 such that $5 \cdot 6 = 30$.

$5 \mid 30$ means "5 divides 30"

Related statements:

If $a \mid b$

- a **divides** b
- a is a **divisor** of b
- a is a **factor** of b
- b is a **multiple** of a

Examples:

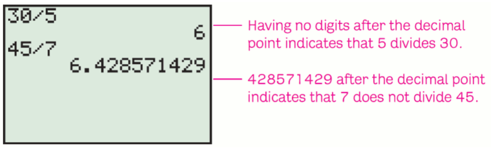
a) Does $7 \mid 21$?

b) Does $2 \mid 3$?

c) Does $5 \mid 15$?

d) What numbers divide 6?

You can use a calculator to test divisibility.



A number is **factored** if it is written as a product of natural numbers.

Examples:

22 = 2 x 11

49 = 7 x 7

100 = 4 x 25 or 10 x 10 or ...

A number bigger than 1 that only has 1 and itself as factors is called a **prime** number.

Smallest examples:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37

There are an infinite number of primes.

Largest known prime:

$2^{57885161} - 1$

Which has 17425170 digits!

← So proud they made a stamp!

A number which is not prime is called **composite**.

Composite numbers have factors other than 1 and themself.

The **Sieve of Eratosthenes** is a method for generating a list of prime numbers.

Skip 1.
Circle 2, cross out multiples of 2.
Go to the next number, it is prime, cross out its muliples.
Repeat.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

If 2 | n, then (n/2) | n.
If 3 | n, then (n/3) | n.
…
If k | n, then (n/k) | n.

What is the largest number to check to see if n is prime?

If k = (n/k), then $k^2 = n$.
So only need to check numbers less than sqrt(n) to see if n is prime.

Prime Numbers

- Example:

Determine whether 83 is prime.

- Solution:

We don't need to check to see if any composites divide 83. Do you see why?
Also, we need not check primes greater than 10:

$$9 < \sqrt{83} < 10.$$

$9^2 = 81$, so 9 is less than the square root of 83.
$10^2 = 100$, so 10 is greater than the square root of 83.

None of the primes 2, 3, 5, and 7 divide 83, so 83 is prime.

Divisibility Tests and Factoring

Number Is Divisible by	Test	Example
2	The last digit of the number is divisible by 2.	2 divides 13,578 because 2 divides 8.
3	The sum of the digits is divisible by 3.	3 divides 21,021 because 3 divides 2 + 1 + 0 + 2 + 1 = 6.
4	The number formed by the last two digits is divisible by 4.	102,736 is divisible by 4 because 4 divides 36.
5	The last digit is 0 or 5.	607,895 is divisible by 5.
6	The number is divisible by both 2 and 3.	802,674 is divisible by both 2 and 3, so it is divisible by 6.
8	The number formed by the last three digits is divisible by 8.	8 divides 230,264 because 8 divides 264.
9	The sum of the digits is divisible by 9.	2,081,763 is divisible by 9 because 9 divides 2 + 0 + 8 + 1 + 7 + 6 + 3 = 27.
10	The number ends in 0.	12,865,890 is divisible by 10.

Divisibility Tests and Factoring

- Example: Test the number 11,352 for divisibility by 6.
- Solution:
  - 11,352 is even, so it is divisible by 2.
  - 1 + 1 + 3 + 5 + 2 = 12, which is divisible by 3, so 11,352 is divisible by 3.
  - Since the number is divisible by both 2 and 3, it is divisible by 6.

Example: Is 201 a prime number?

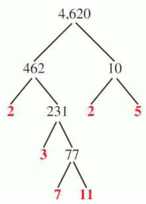
Divisibility Tests and Factoring

THE FUNDAMENTAL THEOREM OF ARITHMETIC Every natural number greater than 1 is a unique product of prime numbers, except for the order of the factors. (Product could mean a single prime number.)

- One way to find the prime factorization of a number is to use a *factor tree*.

Divisibility Tests and Factoring

- Example: Factor 4,620.
- Solution:



$4{,}620 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 2 \cdot 5 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$

Example: Factor 1050

Great Dilemma of our Times:

Hot Dog Buns are sold in packs of 8
Hot Dogs are sold in packs of 10

What is the smallest number of hot dogs in buns that can be made with no leftovers?

DEFINITION The **greatest common divisor** or **GCD** of two natural numbers is the largest natural number that divides both numbers.

Examples:

10 is the GCD of 30 and 70

9 is the GCD of 27 and 900

1 is the GCD of 5 and 11

To find the GCD of two numbers:
- find the prime factorization of both numbers (with factor trees).
- the primes (and their multiples) that they have in common is the GCD.
- if they have nothing in common then the GCD is just 1.

Example: The GCD of 1050 and 768

1050 = 2 x 3 x 5 x 5 x 7

768 = 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 3

Example: The GCD of 220 and 273

220 = 2 x 2 x 5 x 11

273 = 3 x 7 x 13

Example: The GCD of 1944 and 26244

$1944 = 2^3 \times 3^5$

$26244 = 2^2 \times 3^8$

DEFINITION The **least common multiple** or **LCM** of two natural numbers is the smallest natural number that is a multiple of both numbers.

Examples:

210 is the LCM of 30 and 70

2700 is the LCM of 27 and 900

55 is the LCM of 5 and 11

Example with 28 and 42

Multiples of 28:
28, 56, 84, 112, 140, 168, 196, 224, 252, ...

Multiples of 42:
42, 84, 126, 168, 210, 252, 294, …

Common Multiples:

Least Common Multiple:

To find the LCM of two numbers:
- find the prime factorization of both numbers (with factor trees).
- the product of the bigger multiple of each prime is the LCM.

Example: The LCM of 220 and 1672

220 = 2 x 2 x 5 x 11

1672 = 2 x 2 x 2 x 11 x 19

Example: The LCM of 220 and 273

220 = 2 x 2 x 5 x 11

273 = 3 x 7 x 13

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Great Dilemma of our Times:

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Greatest Common Divisors and Least Common Multiples

FINDING THE GCD AND LCM BY USING FACTORIZATION To find the GCD and LCM of two numbers, do this:

1. Factor both numbers, and write each as a product of powers of primes.
2. To calculate the GCD, multiply the *smallest* powers of any primes that are common to both numbers.
3. To calculate the LCM, multiply the *largest* powers of all primes that occur in either number.

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Alternative method using powers

Example: The GCD of 600 and 540

One 3 divides both. One 5 divides both.
$600 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$ and $540 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5.$
Two 2s divide both.

Write $600 = 2^3 \cdot 3^1 \cdot 5^2$ and $540 = 2^2 \cdot 3^3 \cdot 5^1$.

In forming the GCD, we multiply the 2^2, the 3^1, and 5^1, which were the *smallest powers* of the primes that divide both numbers.

So the GCD is $2^2 \cdot 3^1 \cdot 5^1$.

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Alternative method using powers

Example: The LCM of 600 and 540

600 requires three 2s. 540 requires three 3s.
$600 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$ and $540 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5.$
600 requires two 5s.

Write $600 = 2^3 \cdot 3^1 \cdot 5^2$ and $540 = 2^2 \cdot 3^3 \cdot 5^1$.

Then in forming the LCM, we multiply the 2^3, the 3^3, and 5^2, which were the *highest powers* of the primes that divide either number. So, the LCM is $2^3 \cdot 3^3 \cdot 5^2$.

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Greatest Common Divisors and Least Common Multiples

If we look at what happened in the last two examples carefully, we see the following pattern:

$600 = 2^3\ 3^1\ 5^2$
$540 = 2^2\ 3^3\ 5^1$

Multiplying the ☐'s gives the LCM.

Multiplying the ○'s gives the GCD.

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Applying the GCD and LCM

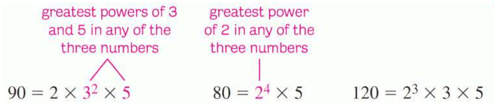
- Example:

Assume that bullet trains have just departed from Tokyo to Osaka, Niigata, and Akita. If a train to Osaka departs every 90 minutes, a train to Niigata departs every 120 minutes, and a train to Akita departs every 80 minutes, when will all three trains again depart at the same time?

(*solution on next slide*)

Applying the GCD and LCM

- Solution:



$2^4 \times 3^2 \times 5 = 720$ minutes = 12 hours